Summary of Classical Mechanics

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All classical systems obey the *principle of least action*:

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt, \tag{1}$$

where S is the action and the trajectory of the system must be such that it takes the minimal value in the time interval $t_2 - t_1$. The quantity $\dot{q}$ is the first time derivative of the position q and it therefore represents the speed. L is the *Lagrangian function* and is defined as the difference between the kinetic energy (expressed in terms of the momentum p)

$$T = \frac{1}{2}\frac{p^2}{m} \tag{2}$$

and the potential energy U expressed in terms of q and whose forms depends on the fields and forces to which to system is subject, i.e.

$$L = T(p) - U(q). \tag{3}$$

Suppose now that there is some variation δq of the function q. The condition of the minimization of the action requires that

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0, \tag{4}$$

that can be rewritten as

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt = 0, \tag{5}$$

where

$$\delta \dot{q} = \frac{d}{dt}\delta q. \tag{6}$$

Integrating by parts the second term we get:

$$\delta S = \frac{\partial L}{\partial \dot{q}} \int_{t_1}^{t_2} \delta q + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0. \tag{7}$$

The first term cancels out because $\delta q(t_1) = \delta q(t_2) = 0$. It only remains the second integral that is zero when the integrand is also zero:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{8}$$

This is the *Lagrange equation* and can be considered as the equation of motion. Considering diverse subsystems j, we can deduce:

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = -\frac{\partial U}{\partial q_j} = \dot{p}_j, \tag{9}$$

and therefore

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \dot{q}_j m_j, \tag{10}$$

in full agreement with Eq. (3). From the last two eqs. we immediately derive

$$m_j \ddot{q}_j = -\frac{\partial U}{\partial q_j}, \tag{11}$$

which, when generalized to a force F, is Newton's second law.

Let us now consider the differential of the Langrange function as a a function of position and speed:

$$dL = \sum_j \frac{\partial L}{\partial q_j} dq_j + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k, \tag{12}$$

which, given Eqs. (9)-(10), can be rewritten as:

$$dL = \sum_j \dot{p}_j dq_j + \sum_j p_j d\dot{q}_j. \tag{13}$$

Since we have the equality

$$d(\sum_j p_j \dot{q}_j) = \sum_j p_j d\dot{q}_j + \sum_j \dot{q}_j dp_j, \tag{14}$$

the last term of Eq. (13) can be rewritten as:

$$\sum_j p_j d\dot{q}_j = d(\sum_j p_j \dot{q}_j) - \sum_j \dot{q}_j dp_j, \tag{15}$$

If we substitute the two terms on the right-hand side (rhs) to the second term on the rhs of Eq. (13) and bring the term $d(\sum p_j \dot{q}_j)$ on the left of the last equation and change all signs, we obtain:

$$d\left(\sum_j p_j \dot{q}_j - L\right) = -\sum_j \dot{p}_j dq_j + \sum_j \dot{q}_j dp_j. \tag{16}$$

Let us now consider the total derivative with respect to time of the Lagrangian function by taking into account of variables p and q only:

$$\frac{dL}{dt} = \sum_j \left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right) . \tag{17}$$

By taking advantage of Eq. (9) we get:

$$\begin{aligned} \frac{dL}{dt} &= \sum_j \left(\dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right) \\ &= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) , \end{aligned} \tag{18}$$

what allows us to write the following equation:

$$\frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \right) = 0. \tag{19}$$

In other words, since the term

$$\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \tag{20}$$

is invariant during the motion, we take it as the definition of the energy of the system, the Hamiltonian, which, thanks to Eqs. (2), (9)-(10), and (16), can be written as:

$$\begin{aligned} H(p,q,t) &= \sum_j p_j \dot{q}_j - L = \sum_j p_j \dot{q}_j - \sum_j \frac{1}{2} p_j \dot{q}_j + U(q) \\ &= T(p) + U(q). \end{aligned} \tag{21}$$

In other words, it is the sum of the kinetic and potential energy, as expected. From the differential equality

$$dH = -\sum_j \left(\dot{p}_j dq_j - \dot{q}_j dp_j \right) \tag{22}$$

we can derive *Hamilton's equations* (that are the equation of motion by making use of the Hamiltonian):

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \text{and} \quad \dot{p}_j = \frac{\partial H}{\partial q_j}, \tag{23}$$

which can also be written in terms of the Poisson brackets as

$$\dot{q}_j = \{q_j, H\}, \qquad \dot{p}_j = \{p_j, H\} . \tag{24}$$

The Poisson brackets for two arbitrary functions f and g are defined as

$$\{f,g\} = \sum_j \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right) \ , \tag{25}$$

and have the following properties

$$\begin{aligned} \{f,g\} &= -\{g,f\} \ , && (26a)\\ \{f,C\} &= 0 \ , && (26b)\\ \{Cf + C'g, h\} &= C\{f,h\} + C'\{g,h\} \ , && (26c)\\ 0 &= \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} \ , && (26d)\\ \frac{\partial}{\partial t}\{f,g\} &= \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} \ , && (26e) \end{aligned}$$

where C, C' are constants and h is a third function. Eq. (26d) is known as Jacobi identity. The advantage of this notation is that, for any function f of q and p, we can write its time evolution as

$$\frac{d}{dt} f = \{f, H\} \ . \tag{27}$$

Let us now write the total derivative of the Hamiltonian (which is the sum of three components):

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_j \frac{\partial H}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial H}{\partial p_j}\dot{p}_j. \tag{28}$$

By making use of Eqs. (23) it is clear that the last two terms on the right disappear, so that we get:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{29}$$

what, in the particular case in which there is no dependence on time, reduces to

$$\frac{dH}{dt} = 0, \tag{30}$$

that can be considered an expression of the law of energy conservation. Since from Eq. (1) we have

$$\frac{dS}{dt} = L, \tag{31}$$

we also have — compare Eq. (9) —

$$\frac{\partial S}{\partial q_j} = p_j. \tag{32}$$

From the latter result the expression

$$\frac{\partial S}{\partial t} = L - \sum_j p_j \dot{q}_j \tag{33}$$

follows which, thanks to Eq. (21), allows us to deduce

$$\frac{\partial S}{\partial t} = -H. \tag{34}$$

The dynamics of a statistical ensemble of classical systems is subjected to the *Liouville equation* (or continuity equation). Let us denote with $\rho(q, p; t)$ the density of representative points that at time t are contained in the infinitesimal volume element $d^n p \, d^n q$ in Γ around q and p. Then it is possible to show that we have

$$\frac{d\rho}{dt} = \{\rho, H\} + \frac{\partial \rho}{\partial t} = 0 \tag{35}$$

or, by applying Eq. (27):

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} \ . \tag{36}$$

Then, the Liouville theorem states that the density of representative points in the phase space Γ is constant, where with *phase space* we denote the Cartesian representation whose coordinates are represented by position and momentum.